Math 255A' Lecture 21 Notes

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1 Continuous Functional Calculus for Self-Adjoint Operators

1.1 Idea for proving the general spectral theorem for self-adjoint operators

Theorem 1.1. *Let T be a self-adjoint operator on H with*

$$a = \inf_{\|x\|=1} \langle Tx, x \rangle, \qquad b = \sup_{\|x\|=1} \langle Tx, x \rangle.$$

Then there exists a spectral family $(E(\lambda))_{\lambda \in \mathbb{R}}$ such that

$$T = \int_{\mathbb{R}} \lambda \, dE(\lambda).$$

This means $\langle Tx, y \rangle = \int_{[a,b]} \lambda \, d\mu_{x,y}$ for all $x, y \in H$, where $\mu_{x,y}$ is the Lebesgue-Stieltjes measure corresponding to $F_{x,y}(\lambda) = \langle E(\lambda), x, y \rangle$.

Method: Consider the map $\mathbb{R}[x] \to \mathcal{B}(H)$ sending $p(t) = \sum_{i=1}^n c_i t^i \mapsto \sum_{i=1}^n c_i T^i$.

Remark 1.1. This is a homomorphism of algebras over $\mathbb{R}$.

The idea is to enrich the domain of this homomorphism to produce many more operators out of T. Why is this relevant? Suppose

$$T = \sum_{i=1}^N \lambda_i E(\lambda_{i-1}, \lambda_i)$$

like in the finite case. Then

$$T^2 = \sum_{i=1}^N \lambda_i^2 E(\lambda_{i-1}, \lambda_i).$$

This generalizes to any polynomial of T. Assume we can do this for the functions $p(t) = \mathbb{1}_{(-\infty,\mu]}(t)$. Then

$$p(T) = \int_a^\mu p(\lambda)\, dE(\lambda) = E(\mu).$$

So this should tell us what $E(\mu)$ is. The proof of the spectral theorem is basically this idea in reverse.

1.2 Continuous functional calculus

When we extend our functional calculus to non-polynomial functions, we only really care what the functions only do on the spectrum of T. In particular, the function $p(T)$ should only depend on the values of p in $[a, b]$, where a, b are as above.

Our next goal is to show that if $p \in R[t]$ and $c \leq p(t) \leq d$ for all $t \in [a, b]$, then $cI \leq p(T) \leq dI$. It is enough to show one side of the inequality, and so it is enough to show it when $c = 0$. So we will show that if $p|_{[a,b]} \geq 0$, then $p(T) \geq 0$.

Lemma 1.1 (sum of squares decomposition)**.** *If $p \in \mathbb{R}[t]$ and $p \geq 0$, then there exist $q_1, \dots, q_m \in \mathbb{R}[t]$ such that $p(t) = \sum_{i=1}^m q_i(t)^2$.*

Remark 1.2. This can actually be shown for $m = 2$.

Proof. Let $p(t) = \sum_{i=0}^n c_i t^i$. If $n = 0$, we are done. Suppose $n \geq 1$ and $p \geq 0$. Then n is even, and $c_n > 0$. Then there exists some $u \in \mathbb{R}$ such that $p(u) = \min p(\mathbb{R}) =: c$. Let $p_1(t) := p(t) - c$. Then $p_1 \geq 0$, and $p_1(0) = 0$. This implies that $(t-u)^2$ divides p in $\mathbb{R}[t]$. so $p_1(t) = (t-u)^2 p_2(t)$. Since p_2 is continuous, $p_2 \geq 0$ with $\deg p_2 = n - 2$. By induction, $p_2(t) = \sum_j q_j(t)^2$, and we get

$$p(t) = \sum_j ((t-u)q_j(t))^2 + (\sqrt{c})^2. \qquad \square$$

Proposition 1.1. *Let T be self-adjoint. If $p|_{[a,b]} \geq 0$, then $p(T) \geq 0$.*

Proof. Step 1: Assume $p \geq 0$. Then $p(t) = \sum_j q_j(t)^2$, so $p(T) = \sum_j (q_j(T))^2$, which is a sum of positive operators: $\langle (q(T))^2 x, x \rangle = \|q(T)x\|^2$.

Step 2: Assume $a = -1$, $b = 1$, so $p|_{[-1,1]} \geq 0$. Let $\varepsilon > 0$, and choose $\delta > 0$ such that $(p+\varepsilon)|_{[-(1+\delta),1+\delta]} \geq 0$. Define $p_n(t) = p(t) = \varepsilon + (\frac{t}{1+\delta})^{2n}$. Then $p_n \geq 0$ for all sufficiently large n. So by case 1, $p_n(T) \geq 0$ for all sufficiently large n. But

$$\|(p+\varepsilon)(T) - p_n(T)\| = \left\| \left(\frac{T}{1+\delta} \right)^n \right\| \leq \frac{\|T\|^n}{(1+\delta)^n} = \frac{1}{(1+\delta)^n} \to 0.$$

So $p_n(T) \xrightarrow{op} (p+\varepsilon)(T)$, which makes $(p+\varepsilon)(T) \geq 0$. Then $p(T) \geq 0$. $\square$

So $p \mapsto p(T)$ satisfies $\|p(T)\|_{\text{op}} \leq \|p|_{[a,b]}\|_{\text{sup}}$. So if $f \in C([a,b])$, define $f(T) = \lim_n p_n(T)$. Then for all $p_n \in \mathbb{R}[t]$ such that $p_n \to f$ uniformly on $[a,b]$ this is well-defined. So $f \mapsto f(T)$ is still a homomorphism: If $p_n \to f$ and $q_n \to g$ in $C([a,b])$, then $p_n q_n \to fg$ in $C([a,b])$. Therefore,

$$(fg)(t) = \lim_n (p_n q_n)(T) = \lim_n p_n(T) q_n(T) = f(T)g(T).$$

This gives us a well-defined functional calculus for continuous functions.

1.3 Weak operator limits of positive functions

Definition 1.1. If $\langle T_n \rangle \in \mathcal{B}(H)$ and $T \in \mathcal{B}(H)$, $T_n \to T$ in the **weak operator topology** if $\langle T_n x, y \rangle \to \langle Tx, y \rangle$.

Remark 1.3. We could also define the **strong operator topology** by $T_n x \to Tx$ for all $x \in H$.

Proposition 1.2. *Suppose $\langle T_n \rangle_n \in \mathcal{B}(H)$ with $T_n \geq 0$ and $T_n \geq T_{n+1}$ for all n. Then there exists a positive $T \in \mathcal{B}(H)$ such that $T_n \xrightarrow{WOT} T$.*

Remark 1.4. We actually get SOT convergence here, but the proof is a bit harder.

Proof. For all $x \in H$, $\langle T_n x, x \rangle$ must converge to some $Q(x,x) \geq 0$. FOr all x, y we get

$$\begin{aligned}
\langle T_n x, y \rangle &= \frac{1}{2}(\langle T_n(x+y), x+y \rangle - \langle T_n x, x \rangle - \langle T_n y, y \rangle) \\
&\to \frac{1}{2}(Q(x+y, x+y) - Q(x,x) - Q(y,y)) \\
&=: Q(x,y).
\end{aligned}$$

Check that $(x,y) \mapsto Q(x,y)$ is symmetric, bilinear, positive, and $Q(x,x) \leq M\|x\|^2$ for some M.

So for each x, the map $Q(x, \cdot) \in H^* = H$ and is bounded: $\|Q(x,\cdot)\| \leq M\|x\|$. By Riesz representation, there exists some $Tx \in H$ such that $Q(x,y) = \langle Tx, y \rangle$ for all x, y. Check that $T \in \mathcal{B}(H)$ is self-adjoint with $T \geq 0$. $\square$

The idea for the next step is to let $\langle f_n \rangle_n \in C([a,b])$ be bounded below with $f_n \downarrow g$ (possible e.g. if $g = \mathbb{1}_{(-\infty,\mu]}$). Then if $f_n \geq f$ in $C([a,b]$, $f_n(T) \geq f_{n+1}(T)$ in $\mathcal{B}(H)$. We will define $g(T)$ as the WOT limit of the $f_n(T)$.